<u>Debt Sustainability Condition – Mathematical Version</u>

The famous debt sustainability condition is often written mathematically as follows:

$$\Delta\left(\frac{\boldsymbol{B_t}}{\boldsymbol{P_t Y_t}}\right) = (\boldsymbol{r_t} - \boldsymbol{g_t})\left(\frac{\boldsymbol{B_{t-1}}}{\boldsymbol{P_{t-1} Y_{t-1}}}\right) + \frac{\boldsymbol{G_t} - \boldsymbol{T_t}}{\boldsymbol{P_t Y_t}} - \frac{\Delta \boldsymbol{H_t}}{\boldsymbol{P_t Y_t}}$$

<u>SIMPLE ANALYTICS:</u>

Public Debt-to-GDP ratio will rise if:

$$\boldsymbol{r_t > g_t}$$

$$\boldsymbol{G_t - T_t > 0}$$

Debt sustainability requires:

$$\Delta\left(\frac{\boldsymbol{B_t}}{\boldsymbol{P_t Y_t}}\right) = \boldsymbol{0}$$

If debt monetization is pursued, $\Delta H_t > 0$, it will help reduce the debt burden. However, debt monetization may result in higher inflation. If we rule out debt monetization ($\Delta H_t = 0$), then debt sustainability requires:

$$-\left(\frac{\boldsymbol{G_t} - \boldsymbol{T_t}}{\boldsymbol{P_t Y_t}}\right) = (\boldsymbol{r_t} - \boldsymbol{g_t})\left(\frac{\boldsymbol{B_{t-1}}}{\boldsymbol{P_{t-1} Y_{t-1}}}\right)$$

or,

$$\left(\frac{\boldsymbol{T_t} - \boldsymbol{G_t}}{\boldsymbol{P_t Y_t}}\right) = (\boldsymbol{r_t} - \boldsymbol{g_t})\left(\frac{\boldsymbol{B_{t-1}}}{\boldsymbol{P_{t-1} Y_{t-1}}}\right)$$

The above condition implies that the government's *primary surplus* (the excess of revenue over non-interest spending) must equal the stock of outstanding public debt to GDP ratio times the difference between the effective real interest rate paid on existing debt and the real GDP growth rate. Otherwise, the ratio of debt to GDP will just explode.

<u>Notation:</u>

G_t: Government Purchases
T_t: Taxes net of Transfers ($T_t = Taxes - Transers$)
$G_t - T_t$: Primary Budget Deficit (note: $G_t > T_t$ implies a primary budget deficit)
B_t: Government Debt in Period *t*
i_t: Nominal Interest Rate
r_t: Real Interest Rate
π_t: Inflation Rate
g_t: Growth Rate of Real GDP
H_t: Monetary Base or High-Powered Money (Currency plus Bank Reserves)
$P_t Y_t$: Nominal GDP
D_t: Overall Budget Deficit

DERIVATION

Reference: *Macroeconomics* (2nd Edition) by R. Glenn Hubbard and Anthony Patrick O'Brien (Publisher: Pearson)

Standard budget deficit is given by:

$$D_t = i_t B_{t-1} + G_t - T_t$$

To capture the role of seignorage (essentially represents a transfer of wealth from individuals holding money to the government), we extend the above equation to include change in monetary base:

$$D_t = i_t B_{t-1} + G_t - T_t - \Delta H_t$$

Note that the change in government debt is given by:

$$\boldsymbol{\Delta B_t = B_t - B_{t-1} = D_t = i_t B_{t-1} + G_t - T_t - \Delta H_t}$$

Divide both sides by nominal GDP and rearrange to get:

$$\boldsymbol{\frac{B_t}{P_t Y_t} = (1 + i_t)\frac{B_{t-1}}{P_t Y_t} + \frac{G_t}{P_t Y_t} - \frac{T_t}{P_t Y_t} - \frac{\Delta H_t}{P_t Y_t}}$$

Modify pervious equation as follows (multiply and divide the first term on the right-hand side by $\boldsymbol{P_{t-1} Y_{t-1}}$):

$$\boldsymbol{\frac{B_t}{P_t Y_t} = (1 + i_t)\left(\frac{P_{t-1} Y_{t-1}}{P_t Y_t}\right)\left(\frac{B_{t-1}}{P_{t-1} Y_{t-1}}\right) + \frac{G_t}{P_t Y_t} - \frac{T_t}{P_t Y_t} - \frac{\Delta H_t}{P_t Y_t}}$$

Note the following condition,

$$\boldsymbol{\left(\frac{P_t Y_t}{P_{t-1} Y_{t-1}}\right) = \frac{P_t}{P_{t-1}}\frac{Y_t}{Y_{t-1}} = (1 + \pi_t)(1 + g_t)}$$

$$\boldsymbol{\left(\frac{P_t Y_t}{P_{t-1} Y_{t-1}}\right) = (1 + \pi_t)(1 + g_t) \approx 1 + \pi_t + g_t}$$

Using the approximation:

$$\boldsymbol{\left(\frac{P_t Y_t}{P_{t-1} Y_{t-1}}\right) \approx 1 + \pi_t + g_t}$$

We get:

$$\boldsymbol{\frac{B_t}{P_t Y_t} = \left(\frac{1 + i_t}{1 + \pi_t + g_t}\right)\left(\frac{B_{t-1}}{P_{t-1} Y_{t-1}}\right) + \frac{G_t}{P_t Y_t} - \frac{T_t}{P_t Y_t} - \frac{\Delta H_t}{P_t Y_t}}$$

Another useful approximation:

$$\left(\frac{\mathbf{1}+\boldsymbol{i_t}}{\mathbf{1}+\boldsymbol{\pi_t}+\boldsymbol{g_t}}\right) \approx \mathbf{1}+\boldsymbol{i_t}-\boldsymbol{\pi_t}-\boldsymbol{g_t}$$

So:

$$\frac{B_t}{P_tY_t} = (1+i_t-\pi_t-g_t)\left(\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\right)+\frac{G_t}{P_tY_t}-\frac{T_t}{P_tY_t}-\frac{\Delta H_t}{P_tY_t}$$

Rearrange terms to get:

$$\Delta\left(\frac{B_t}{P_tY_t}\right) = \frac{B_t}{P_tY_t}-\left(\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\right) = (i_t-\pi_t-g_t)\left(\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\right)+\frac{G_t}{P_tY_t}-\frac{T_t}{P_tY_t}-\frac{\Delta H_t}{P_tY_t}$$

$$\Delta\left(\frac{B_t}{P_tY_t}\right) = (i_t-\pi_t-g_t)\left(\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\right)+\frac{G_t}{P_tY_t}-\frac{T_t}{P_tY_t}-\frac{\Delta H_t}{P_tY_t}$$

Note: $i_t-\pi_t=r_t$

Hence,

$$\Delta\left(\frac{B_t}{P_tY_t}\right) = (r_t-g_t)\left(\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\right)+\frac{G_t-T_t}{P_tY_t}-\frac{\Delta H_t}{P_tY_t}$$